MTH 303

Real analysis **Homework 3**

1. Prove that each of the following sequences $\{x_n\}$ converges to a limit x.
   (a) $x_n = \frac{n+1}{2n+3}$
   (b) $x_n = \frac{n}{n^2-n+1}$
   (c) $x_n = \sqrt{n+1} - \sqrt{n}$
   (d) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$
2. Let $\{x_n\}$ be a sequence of real (or complex) numbers, then $x_n \to 0$ if and only if $|x_n| \to 0$ if and only if $x_n^2 \to 0$.
3. Let $\{x_n\}$ be a sequence of real (or complex) numbers. If $x_n \to x$ then $|x_n| \to |x|$. Also, show that the converse is not true.
4. Let $x_n \to x$ and $y_n \to y$ and $\alpha \in \mathbb{R}$. Then
   (a) $x_n + \alpha y_n \to x + \alpha y$
   (b) $x_n.y_n \to x.y$
   (c) $\frac{1}{x_n} \to \frac{1}{x}$, provided $x \neq 0$.
5. Let $\{x_n\}$ be a sequence such that $x_n \to 0$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Show that the sequence $\{y_n\}$, where $y_n = x_{\sigma(n)}$, converges to 0.
6. (Sandwich lemma) Let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be sequences such that $x_n \leq z_n \leq y_n$, $\forall n$. If $x_n \to x$ and $y_n \to x$, then $z_n \to x$.
7. Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.
8. Prove that $\{x_n\}$ is Cauchy if $|x_n| \leq \frac{n+1}{n^2+n+1}$, $\forall\, n \in \mathbb{N}$
9. Prove that $\{x_n\}$ is Cauchy if $|x_{n+1} - x_n| \leq a^{-n}, \forall\, n \in \mathbb{N}$ for some $a > 1$.
10. Given a sequence of real numbers $\{x_n\}$, consider the sequence of its arithmetic means
$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n},\ n \in \mathbb{N}.$$
    (a) If $\lim_{n\to\infty} x_n = x$, then prove that $\lim_{n\to\infty} s_n = x$.
    (b) Construct a sequence $\{x_n\}$ which does not converge, although $\lim_{n\to\infty} s_n = 0$.
    (c) Can it happen that $x_n > 0$ for all $n > 0$ and that $\limsup x_n = \infty$, although $\lim_{n\to\infty} s_n = 0$.
    (d) Set $a_n = x_{n+1} - x_n,\ n \geq 1$. Show that
$$x_{n+1} - s_n = \frac{1}{n}\sum_{k=1}^{n} k a_k.$$
    Assume that $\lim_{n\to\infty}(n a_n) = 0$ and that $\{s_n\}$ converges. Prove that $\{x_n\}$ converges.

11. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots,$ by the recursion formula
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$
Prove that $\{x_n\}$ is monotonically decreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

12. (a) Let $x_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that the sequence $\{x_n\}$ diverges to ∞.
    (b) Let $\{x_n\}$ be a sequence of positive real numbers and consider $y_n = \sum_{k=1}^{n}(x_k + \frac{1}{x_k})$. Show that the sequence $\{y_n\}$ diverges to ∞.
    (c) Let $\{x_n\}$ and $\{y_n\}$ be sequences of positive real numbers. Assume that $\lim_{n\to\infty} \frac{x_n}{y_n} = \alpha > 0$. Show that $\lim_{n\to\infty} x_n = \infty$ if and only if $\lim_{n\to\infty} y_n = \infty$.

13. Let $\{x_n\}$ be a sequence of positive real numbers. Assume that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha > 0$. Then show that $\lim_{n\to\infty}(x_n)^{\frac{1}{n}} = \alpha$.

14. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of real numbers. Show that
    (a) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$
    (b) $\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n$
    (c) If $x_n \to x$, then
        i. $\limsup(x_n + y_n) = x + \limsup y_n$
        ii. $\liminf(x_n + y_n) = x + \liminf y_n$
    (d) If $x_n \to x$ and $x \geq 0$ then
        i. $\limsup(x_n y_n) = x \limsup y_n$
        ii. $\liminf(x_n y_n) = x \liminf y_n$

15. Find $\limsup x_n$ and $\liminf x_n$ for each of the following sequences
    (a) $x_n = (-1)^{n+1}$
    (b) $x_n = (-1)^n + \frac{1}{n}$
    (c) $x_n = \frac{1}{n} + \frac{(-1)^n}{n^2}$
    (d) $x_n = (1 + \frac{1}{n})^n$